Engineering Management BMEVITMMB03

Game theory and its applications

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Basics of game theory

- Players
- „Strategy”
- payoff
- Laws of game

Historical overview

- **John von Neumann:**
  minimax theorem
- **John Nash:**
  noncooperative game theory
- **John Harsányi:**
  complete information games
- **Lloyd S. Shapley:**
  cooperative theory

Parts of game theory

Noncooperative games

- Competition
- No communication

Examples:

- card games
- board games

Cooperative games

- Cooperation and competition
- Communication

Examples:

- team games

Simple dilemmas

- Prisoner's dilemma
- Iterated prisoners' dilemma (IPD)
- Tragedy of the Commons

Simple dilemmas

- **Prisoner's dilemma**

	Admit		Deny	
Admit	5	5	0	10
Deny	10	0	1	1

Noncooperative game theory

- Perfect information vs. imperfect information
- Zero-sum vs. non-zero-sum
- two vs. more players

Noncooperative matrix games

$$\underline{\underline{A_1}} + \underline{\underline{A_2}} = \underline{\underline{0}} \quad \Rightarrow \quad \underline{\underline{A_2}} = -\underline{\underline{A_1}}$$

	1. strategy	2. strategy		n. strategy	
1. st.	a_{11}	a_{12}	…	a_{1n}	
2. st.	a_{21}	a_{22}	…	a_{2n}	
…					
m. st.	a_{m1}	a_{m2}	…	a_{mn}	

a_{ij} are the payoffs, where i denotes the strategy of first player, and j denotes the strategy of second player.

Noncooperative matrix games (1.)

$$\underline{\underline{A_1}} + \underline{\underline{A_2}} = \underline{\underline{0}} \quad \Rightarrow \quad \underline{\underline{A_2}} = -\underline{\underline{A_1}}$$

	1. strategy	2. strategy		n. strategy	
1. st.	a_{11}	a_{12}	…	a_{1n}	**$b_1 = \min \{a_{1j}\}$**
2. st.	a_{21}	a_{22}	…	a_{2n}	**$b_2 = \min \{a_{2j}\}$**
…					
m. st.	a_{m1}	a_{m2}	…	a_{mn}	**$b_m = \min \{a_{mj}\}$**
					$\max \{b_i\} = \max \{\min \{a_{ij}\}\}$

a_{ij} are the payoffs, where i denotes the strategy of first player, and j denotes the strategy of second player.

Noncooperative matrix games (2.)

$$\underline{\underline{A_1}} + \underline{\underline{A_2}} = \underline{\underline{0}} \quad \Rightarrow \quad \underline{\underline{A_2}} = -\underline{\underline{A_1}}$$

	1. strategy	2. strategy		n. strategy	
1. st.	a_{11}	a_{12}	…	a_{1n}	
2. st.	a_{21}	a_{22}	…	a_{2n}	
…					
m. st.	a_{m1}	a_{m2}	…	a_{mn}	
	$c_1=\max\{a_{i1}\}$	**$c_2=\max\{a_{i2}\}$**	**…**	**$c_n=\max\{a_{in}\}$**	**$\min\{c_j\} = \min\{\max\{a_{ij}\}\}$**

a_{ij} are the payoffs, where i denotes the strategy of first player, and j denotes the strategy of second player.

Saddle point

Condition of existing saddle point:

$$\operatorname*{Max}_{i=1}^{m}\{\operatorname*{Min}_{j=1}^{n}\{a_{ij}\}\}=\operatorname*{Min}_{j=1}^{n}\{\operatorname*{Max}_{i=1}^{m}\{a_{ij}\}\}$$

Optimal strategy of 1. player and 2. player is denoted by i^0, and j^0, respectively. Payoff value is the value of the game : $v = a_{i^0 j^0}$, and (i^0, j^0, v) triple is the solution of the game.

Equivalence and Interchangeability

- **Equivalence**: If there are more saddle points – e.g. (i^0, j^0) and (i^1, j^1) –, then the payoff values of them are equal, i.e.: $a_{i^0 j^0} = a_{i^1 j^1}$
- **Interchangeability**: furthermore the (i^0, j^1) and (i^1, j^0) point are saddle points as well.

Fundamental Theorem of Game Theory

- **Pure – Mixed strategies**
- **Fundamental Theorem of Game Theory: Every matrix game with mixed strategy has solution.**

Noncooperative bimatrix game

- At non-zero-sum game the matrixes are independent and these can be different: **bimatrix game**.
- Average payoffs of two players at mixed strategy:

$$E_1 = p^T \cdot A \cdot q \qquad E_2 = p^T \cdot B \cdot q$$

where p and q are the probability vectors of the two players.

Nash equilibrium

A strategy pair (p^0, q^0) is Nash equilibrium if:

$$E_1(p^0, q^0) \geq E_1(p, q^0) \qquad \forall p$$

$$E_2(p^0, q^0) \geq E_2(p^0, q) \qquad \forall q$$

Solvability

A noncooperative bimatrix game is solvable (according to Nash equilibrium) if every equilibrium possesses attributes of equivalence and interchangeability.

The value of the game: $u^0 = E_1(p^0, q^0)$, $v^0 = E_2(p^0, q^0)$.

The solution of the game: (p^0, q^0, u^0, v^0).

Dominance

A strategy pair *A* (p^1, q^1) dominates strategy pair *B* (p^2, q^2) at a bimatrix game, if

$$E_1(p^1, q^1) \geq E_1(p^2, q^2)$$

$$E_2(p^1, q^1) \geq E_2(p^2, q^2)$$

Dominance-solvable game

The iterated elimination of dominated strategies is one common technique for solving games that involves iteratively removing dominated strategies.

Problem: What is the situation when the game has equilibrium and this is dominated by another point?

Cooperative game theory

- Set of players: $N = \{1, . . . , n\}$.
- The group of players $S \subseteq N$ is the coalition. Some special cases:
  - N is grand coalition,
  - $\emptyset$ is empty coalition.

Characteristic function

- **characteristic function : v(S)**
  consists the maximal payment for each S.
- In a game (N, v)
  the imputation $x = (x_1, \ldots, x_n)$ is
  - **feasible** for S coalition if
    $$\sum_{i \in S} x_i \le v(S)$$
  - **acceptable** for S coalition if
    $$\sum_{i \in S} x_i \ge v(S)$$

Distribution, core

In a game (N, v) the **imputation** x = (x_1, . . . , x_n) is

- **efficient** if $$\sum_{i \in N} x_i = v(N)$$
- **individual rational** if $$x_i \geq v(\{x_i\})$$
- **coalitional rational** if this is an efficient and for every S $$\sum_{i \in S} x_i \geq v(S)$$

Shapley value

At N players in a game $v \in G^N$ the i^{th} player's Shapley value is

$$\varphi_i(v) = \sum_{S \subseteq N \setminus i} \frac{|S|!(|N \setminus S| - 1)!}{|N|!} \left(v(S \cup i) - v(S) \right)$$

Given any "ordering" of the players, where each order is equally likely, the **Shapley value** ϕ_i measures the expected marginal contribution of player *i* over all orders to the set of players who precede her.

Game theory with engineering applications

- Strategic analysis of transition IPv4 – IPv6
- Advertisement strategy of service-quality attributes
- Network neutrality
- Cooperation in self organizing networks
- Energy housekeeping of mobile devices
- Transportation systems
- Frequency auction

Frequency auction

Based on number of biddings

- first-price sealed-bid auction
- dynamic (open) auction

Based on items

- Single-unit auctions
- Multiunit auctions